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Topology Optimization
Using Mixed Finite Elements

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Chapter 1

Introduction and motivation

Topology optimization is nowadays a fertile area of research concerned with the topical issue of defining the best design that solves an assigned physical problem with prescribed performance requirements or other kind of restraints. This general concept may be applied to different practical contexts and industrial applications, where, already in an early stage of the design process, questions as finding an optimal lay-out topology in terms of static or dynamic stiffness, cost, assigned structural performances and so on must be affordably answered. Since the pioneering paper [13] where the topology optimization concept was introduced as an innovative and powerful approach to structural design, many steps have been taken in several directions. The *legalization* of one of the most used material interpolation laws, the SIMP (Solid Isotropic Material with Penalization) approach [15], from the standpoint of constitutive theory appears to be a key step toward the acceptance of topology optimization within the designers community. Furthermore, a few recent results on the existence and uniqueness of the solution to the optimal design problem have provided the entire formula-

tion with a sound mathematical basis [98]. Topology optimization may be considered as a mature discipline especially from the point of view of the applications, since new grounds have been explored, moving from the original field of structural design towards new branches as material design or multiphysics problems, even if many of the early topics remain quite actual and still open, in view of innovative affordable solutions.
This development, achieved over more than twenty years of research and applications, is not only made of advances in mathematics, technology and material mechanics, but is also due to the efforts made within the field of the so called computational sciences. Many interests have been in fact directed towards this topic with the aim of providing numerical instruments able to solve the computational difficulties concerned with the optimal topology formulations and to successfully menage large–scale problems. An extensive literature has been produced regarding minimizer algorithms used to solve the optimum problem, i.e. CONLIN [49] or MMA [137], or referring to the solution of the numerical instabilities such as the *checkerboard phenomenon* [130]. One of the crucial aspects of the more commonly used solving methodologies concerns the finite element approach. Most of the works refer to the adoption of displacement–based finite element methods, while not so many other discretization strategies have been so far investigated, with the exception of the interests recently directed to non–conforming finite elements [65] or displacement–pressure discretizations [129].

Within such a scenario, the aim of this work is to propose alternative formulations for topology optimization by distribution of isotropic material, relying on bidimensional mixed finite element schemes and exploiting the benefits that these methods may provide on several topics of the optimal design discipline.

The variational principle of Hellinger–Reissner is herein used and two dual formulations presented and discretized. The first one is simpler and uses classical polynomial finite elements to approximate a regular displacement field and a piecewise discontinuous stress field. Conversely, the dual formulation, often referred to as *truly–mixed* in the literature [21], interpolates displacements with discontinuous functions while regular ones are used for the stresses. The adoption of these finite element techniques within a topology optimization framework has direct consequences on the numerical stability and on convergence features of the method. These issues are firstly investigated with special regard to the checkerboard problem and the adopted discretizations for the density field. Except for this technical aspect, mixed schemes have two important properties that may be usefully exploited in a topology optimization context, i.e.:

- the accuracy in the evaluation of stresses, due to the additional discretization of the stress field, that does not call for any post-processing technique typical of displacement-based finite elements;
- the capability of passing the inf-sup condition, for the "truly–mixed" formulation, even in the case of incompressible material, feature that is not shared by commonly used displacement-based discretizations.

In the present work the first feature is exploited to deal with the still open problem of finding the optimal topology with local stress constraints on material strength [46]. Managing this topic the so–called *singularity problem* [117], a numerical trouble that often prevents form convergence to expected

global minima, is also faced and a novel methodology, referred to as *qp*–approach, is presented and analyzed within the proposed formulations with the aim of providing an alternative solution to the problem.
The second of the above two main features is conversely exploited in the sequel to find optimal designs for incompressible materials, providing the numerical robustness needed to handle the incompressibility property within an optimization context. Since these materials have recently been used in different applications mainly concerned with vibration issues and aseismic design, alternative topology optimization formulations are presented and tested not only in a static framework but also within eigenvalue–based methodologies for dynamic design. The capability of the *truly–mixed* method to pass the *inf–sup condition* even in presence of incompressible material is moreover used to model fluid phases with the aim of solving pressure–load problems, moving from the approach recently proposed in [129]. The accuracy in the evaluation of the stress field is exploited in this context to propose an alternative methodology against the achievement of final designs that present cavities whose boundaries are acted upon by hydrostatic pressure. This kind of topologies may be in fact of no practical use within certain applications.

The outline of the work is as follows. Chapter 2 introduces the basic concepts of the topology optimization discipline, presenting the state of the art in terms of methods and applications further dealt with, in the sequel of the work. Chapter 3, after a first insight on fundamentals of the finite element method, is mainly concerned with theoretical and numerical issues related to mixed finite elements, deriving both the continuous and the discrete forms of the dual formulations that descend from the variational principle of Hellinger–Reissner. Chapter 4 tackles the checkerboard problem and other numerical issues related to the adoption of the introduced mixed finite elements formulations along with alternative choices in terms of density discretizations. In this regard, preliminary optimizations that maximize the stiffness of the structures are performed, according to the classical framework of topology optimization for minimum compliance. Chapter 5 and Chapter 6 are concerned with the research of optimal topology with local stress constraints. The first of the two chapters refers to the singularity problem, presenting the qp–approach, its features and numerical comparisons with respect to the classical ε–relaxation [35], that is the traditional methodology applied to overcome the numerical difficulties related to the arising of the stress *singularity phenomenon* at zero density. The second one is mainly concerned with the implementation of stress constraints that exploit both the mixed finite element discretizations within a minimum compliance optimization framework. Different optimal designs achieved with and without stress constraints are presented and analyzed from the point of view of their mechanical behavior. Peculiar attention is moreover paid in this context to the topologies obtained by means of the truly-mixed discretization in comparison with those found by the dual mixed setting. Chapter 7 presents the topology optimization of incompressible media for maximum stiffness, along with the numerical difficulties that may be encountered in the research of pure 0–1 designs under plane strain conditions. Relevant material interpolation strategies are introduced and tested to solve this problem. Furthermore, different families of designs are presented, comparing plane strain and plane stress conditions for cases of compressible and incompressible material design, thus pointing out peculiar mechanical differences related to the incompressibility feature. Chapter 8 exploits the capability of handling topology optimization of incompressible materials to propose alternative eigenvalue–based formulations that may be used to deal, within a simplified setting, with typical aseismic isolations problems, as the preliminary design of bi–material aseismic bearing devices for which a suitable multi–phase material interpolation is derived. Chapter 9 implements the truly–mixed discretization within the approach originally introduced in [129] that is based on the modeling of a phase of fluid material in order to cope with pressure–load problems. An alternative technique, exploiting the imposition of stress constraints on the incompressible phase, is herein proposed and tested to avoid the achievement of final designs that present undesired cavities filled with fluids. Chapter 10 summarizes the work, pointing out the main issues and results discussed in the previous chapters and presenting ideas for future developments.

Chapter 10

Conclusions

The work has addressed the issue of introducing and exploiting the adoption of mixed finite elements for plane linear elastic problems within the framework of topology optimization by distribution of isotropic material.

As detailed in Chapter 2, the topology discipline is a relatively recent but well–established research field that provides designers with numerical procedures having the aim of achieving optimal designs for several applications. A crucial aspect of the methodology relies on the choice of the finite elements schemes used in the discretization of the fields involved in the solution of the elasticity equation. With respect to this subject, most of the traditional approaches rely on classical displacement–based techniques and not so many alternatives have been so far investigated.

To this purpose, Chapter 3 has introduced the variational principle of Hellinger–Reissner that has been herein exploited to derive two dual weak formulations of the elasticity problem. The first one uses classical polynomial finite elements to approximate a regular displacement field and a piecewise discontinuous stress field. The second one, often referred to as *truly–mixed*, conversely interpolates displacements with discontinuous functions while stresses with regular ones. Both the formulations have been implemented in discretized forms, having the aim of pointing out the main benefits of the mixed approach with respect to classical displacement–based finite elements. The independent interpolation of stresses, according to the degree of approximation of the adopted shape functions, provides in fact an increased accuracy in the evaluation of the relevant stress field but also of the displacement one. Furthermore, a few mixed approximations are able to pass the *inf–sup condition* even in case of incompressible material thus providing a robust analysis tool that does not encounter the well–known *locking phenomenon*. Both these advantages are shared by the herein implemented discretization of the *truly–mixed* formulation, based on the Johnson Mercier composite triangle.

Within a topology optimization framework, the coupling of displacement (and stress) approximations with density interpolation schemes may generate or resolve numerical instabilities of the procedure. Chapter 4 has addressed this topic, presenting and investigating the dual frameworks that exploit the above cited mixed finite element schemes. While the first one has shown features that are similar to equivalent displacement–based approach, the truly–mixed setting has not experienced the well–known *checkerboard problem*, when coupled with an element–based discretization. However, not to increase the computational burden tied to the adoption of mixed schemes, the nodal–based density setting has been shown to be an affordable choice for the appropriate description of the layout of final designs, exploiting moreover peculiar benefits on the issue of length scale control. Within the classical framework of topology optimization for minimum compliance problem the *truly–mixed* scheme has shown an high accuracy of the achieved results and robustness against instabilities.

The numerical assessment of these basic features has therefore allowed the extension of the mixed optimization framework towards the exploitation of the benefits peculiar to the finite element schemes, in order to deal with stress–constrained problems and topology optimization involving incompressible media.

Preliminarily to the former issue, Chapter 5 has addressed the delicate convergence difficulty that affects stress constraints imposition, i.e. the *singularity phenomenon*. For the solution, an alternative method, called *qp*–approach, has been herein introduced and compared to the classical ε–relaxation. Among its features, the peculiar advantage consists in the smoothness of the manipulation introduced on constraints equations, that improves convergence features and does not involve full density range, thus eventually allowing for non–iterative design procedures.

The methodology has been largely exploited in Chapter 6, where the minimum compliance setting based on mixed finite elements has been applied in conjunction with a set of local stress constraints. The investigations have pointed out the remarkable differences that may be found comparing designs achieved with and without the inclusion of stress constraints. Peculiar attention has been paid to the description of the implemented imposition of the local stress requirements, that gain in accuracy and numerical tractability due to the independent interpolation of the stress field. A suitable methodology has been furthermore developed in order to exploit the accuracy and fast convergence of JM–based approach in stress constraints enforcement, without paying the expected increase in terms of computational cost.

The methodology, that involves only the average degrees of freedom of the adopted truly–mixed discretization, has been shown to produce feasible results that exhibit an improved mechanical behavior with respect to the ones achieved by the dual less accurate approximation.

To exploit the capability of JM discretization to robustly deal with the incompressibility feature, Chapter 7 has straightforwardly extended the minimum compliance *truly–mixed* framework to the optimization of such kind of materials. Within this class of problems, classical interpolation laws that do not penalize the incompressibility feature in the low density range may generate final designs where undesired grey regions takes full advantage of this property, under plane strain conditions. An alternative penalization law for stiffness has been therefore proposed and tested to assess its capability for achieving pure 0–1 designs. Furthermore, several examples have been studied under plane strain and plane stress conditions, for cases of compressible and incompressible material. This comparative investigation has mainly pointed out that remarkably different designs may be achieved within the plane strain incompressible case. Under these conditions the material tends to assume in fact a layout that efficiently exploits the significant increase in the volumetric stiffness, when acted upon by isotropic stress states.

Classical frameworks for the maximization of the first eigenvalue have been addressed in Chapter 8, in order to provide optimization techniques for incompressible media also within a dynamic setting. The problem of *localized modes* has been firstly tackled by means of the introduction of an alternative mass interpolation, that assures an appropriate mass–to–stiffness ratio all over the density range. Having the aim of dealing with the preliminary de-

sign of isolation devices, whose manufacturing involves incompressible materials as rubber–likes ones, a novel eigenvalue–based formulation has been furthermore proposed, based on the achievement of the maximal vertical stiffness with additional requirements on the horizontal dynamic flexibility. The above methodology has been applied to the simplified design of devices made of both steel and rubber–like material, relying on an alternative bi–phase material law.
The capability of handling incompressible media has been furthermore exploited in Chapter 9 to perform the topology optimization for pressure–load problems basing on a method that involves the presence of a fluid phase. To this purpose an alternative "bi–material with void" interpolation has been introduced and the robustness of the JM element in the evaluation of both displacements and stresses has been herein used to cope with the problem of filled cavities, i.e. the arising of internal holes whose boundaries are acted upon by hydrostatic pressure. The imposition of a set of suitable pressure constraints has shown in fact to be able to overcome this possible problem without resorting to more demanding traditional procedures.

The above applications have outlined that mixed methods, with peculiar reference to the *truly–mixed* formulation, may be usefully exploited in topology optimization. The accuracy in the imposition of stress–constraints and the robust handling of incompressible materials are in fact peculiar benefits to the introduced formulations with respect to traditional displacement–based schemes.
A possible drawback of the method is tied to the increased computational burden related to the independent interpolation of stresses. To this pur-

pose suitable algorithms and implementation techniques have been adopted throughout the work to reduce the CPU–times. The most demanding procedure encountered in the simulations was not however tied to the finite element scheme, but, rather, to the local imposition of stress constraints within the minimization procedure. A possible way out to the problem could consist in the adoption of adaptive meshing techniques [150] in the topology optimization framework. This is expected to allow a finer discretization of the zones that experiences the higher stress concentration and, at the same time, a remarkable reduction in the overall number of local constraints.
The peculiar benefits of the mixed methods have been exploited and investigated in this work within a linear elastic bidimensional context. Challenging developments of the presented procedures include therefore the extension to geometrical and constitutive non–linearities along with the implementation of three–dimensional approaches. While the latter argument is mainly concerned with the remarkable increase in the number of unknowns expected in 3D problems, the former is in fact very tricky to be dealt with, because of the complexity of more subtle numerical and computational issues with respect to both the optimization framework and the finite element schemes. The main difficulty is maybe the setting of affordable sensitivities computations in the non–linear framework, as pioneered in the already cited works by [28] and [125]. Furthermore, additional stability issues of mixed finite elements in large strain analysis for rubber–like solids have to be dealt with, concerning the delicate discretization of the incremental version of the Hellinger–Reissner variational principle, see i.e. [92].
Recent contributions have outlined peculiar benefits that may be derived from the assumption of linear elastic model embedded in the optimization

procedures that are alternative to the classical Cauchy setting. Topology investigations have been already performed basing on the micro–polar Cosserat solid [113], but also other multi–field theories [32] should be investigated for structural purposes, as micro–cracked models [85]. An alternative way to cope with fractured media involves non–linear procedures that handle crack propagation taking full advantage from the well–suited nature of the discretizing fields within the *truly–mixed* approach. Discontinuity of displacements and regularity of stress fluxes seem ideally tailored to deal with cohesive fracture models, as mentioned in Section 3.8.3. Having the aim of exploiting these features in optimal design procedures, current investigation is mainly concerned with the assessment of the numerical setting of this analysis instrument [26], also including stochastic effects [23].